Chapter 8

Radicals

Section 8.1
Simplifying Radical Expressions

A radical expression is one that contains roots. The number under the radical sign is called the radicand.

$$\sqrt{81}$$

All positive, real numbers have roots, but negative numbers do not.
The *perfect squares* such as 4, 9, 16, 25, 36... all have roots that are whole numbers:

$$\sqrt{4} = 2$$
$$\sqrt{9} = 3$$
$$\sqrt{16} = 4$$
$$\sqrt{25} = 5$$
$$\sqrt{36} = 6$$

and all other positive numbers in between such as 2, 3, 5, 6, 7, 8, 10, 11... also have roots, but they are irrational numbers:

$$\sqrt{2} = 1.4142...$$
$$\sqrt{3} = 1.7320...$$
$$\sqrt{5} = 2.2360...$$
$$\sqrt{x^2} = x$$
$$\sqrt{y^4} = y^2$$

Negative numbers do not have roots. A **negative** root, multiplied by its identity, another **negative** root, produces a **positive** number ($-2 \times -2 = +4$). Thus, the square root of positive numbers can be both, positive and negative, leaving negative numbers without square roots.

Example: If $3 \times 3 = 9$ and $-3 \times -3 = 9$

then the square root of 9 can be both +3 and –3 $\sqrt{9} = \pm 3$

and the square root of $\sqrt{-9}$, for example, cannot be found.

Practice:

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RADICAL EXPRESSIONS

A radical expression, also called a radicand, is any expression found under a radical ($\sqrt{x}$). Moreover, negative outcomes of radical expressions are not real numbers.

Example: What values of x will make $\sqrt{x-5}$ a real number?

When $x = 0, 1, 2, 3$ or 4, the radical is not a real number because the result is a negative radicand. However, x values of 5 or above are possible.

When $x = 0$ the radical is $\sqrt{-5}$ no answer because the radicand is negative
When $x = 1$ the radical is $\sqrt{-4}$ no answer because the radicand is negative
...
When $x = 4$ the radical is $\sqrt{-1}$ no answer because the radicand is negative
When $x = 5$ the radical is $\sqrt{0} = 0$ (first real number)

Answer: $x \geq 5$

Example: What values will make $\sqrt{x^2+2}$ a real number?

Because squaring a negative number gives always a positive answer, in the example above all real numbers (including all negative numbers) are values that would make $\sqrt{x^2+2}$ a real number.

Example: What values will make $\sqrt{x^2}$ a real number?

$$\sqrt{x^2} = |x| \quad (x \text{ could be any real number})$$

Example: What values will make $\sqrt{5x+7}$ a real number?

$$5x + 7 \geq 0$$
$$5x \geq -7$$
$$x \geq -\frac{7}{5}$$

Answer: $x \geq -\frac{7}{5}$

Example: What values will make $\sqrt{(x-3)^2}$ a real number?

$$\sqrt{(x-3)^2} = |x-3| \quad (x \text{ could be any real number})$$

Example: What values will make $\sqrt{\frac{1}{9}y^2}$ a real number?

$$\sqrt{\frac{1}{9}y^2} = \frac{1}{3}|y| \qquad (y \text{ could be any real number})$$

Example: What values will make $\sqrt{4x^2 - 12xy + 9y^2}$ a real number?

Factor first: $\sqrt{4x^2 - 12xy + 9y^2} = \sqrt{(2x - 3y)(2x - 3y)}$

$$\sqrt{(2x - 3y)(2x - 3y)} = |2x - 3y| \qquad (x \text{ and } y \text{ could be any real number})$$

Practice:

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SIMPLIFICATION OF RADICALS (for nonnegative real numbers)

To simplify a radical means to reduce the radical to the point of not having any perfect squares represented in the radicand. In other words, every two of the same number—or base—under the radical sign, represents one outside the radical sign, the rest stays under the radical.

Example: Simplify $\sqrt{18}$

Use prime factorization to simplify radical $\sqrt{18} = \sqrt{2 \times 3 \times 3}$ (simplify 3^2)

Answer: $3\sqrt{2}$

Example: Simplify $\sqrt{16b^2}$

Use prime factorization to simplify radical $\sqrt{16b^2} = \sqrt{4 \times 4 \times b \times b} = 4b$

Perfect square, radical sign gone

Example: Simplify $\sqrt{25b}$

Use prime factorization to simplify radical $\sqrt{25b} = \sqrt{5 \times 5 \times b} = 5\sqrt{b}$

Example: Simplify $\sqrt{125a^3}$

Use prime factorization to simplify radical $\sqrt{125a^3} = \sqrt{5 \times 5 \times 5 \times a \times a \times a} = 5a\sqrt{5a}$

Example: Simplify $\sqrt{y^7}$

Use prime factorization to simplify radical $\sqrt{y^7} = \sqrt{yyyyyyy} = y^3\sqrt{y}$

Example: Simplify $\sqrt{3a^2 + 30a + 75}$

Factor the 3 first $\sqrt{3(a^2 + 10a + 25)}$

Simplify the perfect trinomial square $\sqrt{3(a+5)(a+5)} = (a+5)\sqrt{3}$

Practice:

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Section 8.2
Operations with Radical Expressions

MULTIPLYING RADICALS
Radicals of the same root may be multiplied under the same radical or separated under different radicals.

Example: Simplify $\sqrt{2} \times \sqrt{6}$

$$\sqrt{2} \times \sqrt{6} = \sqrt{2 \times 6} = \sqrt{12} = \sqrt{2 \times 2 \times 3} = 2\sqrt{3}$$

Example: Simplify $\sqrt{18} \times \sqrt{24}$

$$\begin{aligned} \sqrt{18 \times 24} &= \sqrt{2 \times 3 \times 3 \times 2 \times 2 \times 2 \times 3} \\ &= 4 \times 3\sqrt{3} \\ &= 12\sqrt{3} \end{aligned}$$

Practice:

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DIVIDING RADICALS

Just like in regular division, radicals must be simplified to complete the answer.
Radicals of the same root may be divided under the same radical or separated under different radicals.

Example: $\frac{\sqrt{24}}{\sqrt{6}} = \sqrt{\frac{24}{6}} = \sqrt{4} = 2$

Example: $\frac{\sqrt{98}}{\sqrt{8}} = \sqrt{\frac{98}{8}}$ reduce fraction by 2 $\sqrt{\frac{49}{4}} = \frac{7}{2}$

A rational expression (fraction) with a radical in the denominator has not been completely simplified. To simplify it, multiply the fraction by **one**, using a ratio made up only by the number or expression of the denominator.

Example: $\sqrt{\frac{3}{x}} = \frac{\sqrt{3}}{\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{3x}}{x}$

Notice the original was multiplied by **one** using the term of the denominator: $\frac{\sqrt{x}}{\sqrt{x}}$. Notice also that, because it is an operation of "simplification", the *value* of the rational number has not changed. In the example immediately below, the value of the starting fraction, $\frac{\sqrt{54}}{\sqrt{12}} = 2.121\ldots$, is also the value of the ending fraction $\frac{3\sqrt{2}}{2} = 2.121\ldots$, which has been simplified.

Example: $\frac{\sqrt{54}}{\sqrt{12}} = \sqrt{\frac{54}{12}}$ reduce fraction $\sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

Example: Simplify $\sqrt{\frac{5a}{7b^3}}$ $\sqrt{\frac{5a}{7b^3}} = \frac{\sqrt{5a}}{\sqrt{7b^3}} \times \frac{\sqrt{7b^3}}{\sqrt{7b^3}} = \frac{\sqrt{35ab^3}}{7b^3} = \frac{b\sqrt{35ab}}{7b^{\cancel{3}2}} = \frac{\sqrt{35ab}}{7b^2}$

Practice:

ADDITION AND SUBTRACTION OF RADICALS

In addition and subtraction, radicals are treated like bases or variables: We only add or subtract the radicals with the same value and root (radicand).

Example: Add $3\sqrt{2} + 5\sqrt{2}$

$$3 + 5 = 8$$

$$3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$$

the radical does not change, add both

Example: Add or subtract $\sqrt{3} + 4\sqrt{3} - 2\sqrt{7}$

$$1 + 4 = 5$$

$$\sqrt{3} + 4\sqrt{3} - 2\sqrt{7} = 5\sqrt{3} - 2\sqrt{7}$$

radical 7 is different, do not subtract

Example: Add or subtract $6\sqrt{5} - 10\sqrt{5} + \sqrt{5}$

$$6 - 10 + 1 = -3$$

$$6\sqrt{5} - 10\sqrt{5} + \sqrt{5} = -3\sqrt{5}$$

all radicals are the same, compute all

Practice:

Section 8.3
Radical Equations

Similarly to the way any equation is solved, an equation with radicals is solved by using the inverse property. In this case, the inverse of a radical is found by raising the radical to a like power. Therefore, the inverse of the **square root** is the **square**.

Example: Solve $\sqrt{x} = 5$

square both sides $(\sqrt{x})^2 = 5^2$

$x = 25$

Example: Solve $\sqrt{3x} - 5 = 13$

move –5 $\sqrt{3x} - 5 + 5 = 13 + 5$

$\sqrt{3x} = 18$ Always isolate radical to the left before squaring.

square both sides $(\sqrt{3x})^2 = 18^2$

divide by 3 $3x = 324$

$x = 108$

Example: Solve $\sqrt{3x-5} = \sqrt{2x+8}$

square both sides $(\sqrt{3x-5})^2 = (\sqrt{2x+8})^2$

$3x - 5 = 2x + 8$

solve for x $3x - 2x = 5 + 8$

$x = 13$

Example: Solve $3\sqrt{x-4} = 5\sqrt{6-x}$

square both sides $(3\sqrt{x-4})^2 = (5\sqrt{6-x})^2$

$9(x-4) = 25(6-x)$

$9x - 36 = 150 - 25x$

$9x + 25x = 150 + 36$

$34x = 186$

$x = 5.47$ (rounding to the nearest hundredth)

Example: Solve $\sqrt{3x^2 + 24x - 18} = 3$

square both sides $(\sqrt{3x^2 + 24x - 18})^2 = 3^2$

subtract 9 from both sides $3x^2 + 24x - 18 = 9$

divide equation by common factor 3 $3x^2 + 24x - 27 = 0$ ➡ $x^2 + 8x - 9 = 0$

Equation is now factored: Which two integers give a product of –9 and a difference of +8?

Only two possible multiplications yield a 9: 9×1 and 3×3, and only $9 - 1 = 8$

Factoring $(x + 9)(x - 1) = 0$ $\qquad x + 9 = 0 \qquad x - 1 = 0$

$\qquad\qquad x = -9 \qquad\qquad x = 1$

EQUATION WITH RADICALS THAT YIELD NO VALUE
Some radical equations cannot be solved.

In examples

$$\sqrt{x} = -8 \qquad -\sqrt{x} = 8 \qquad \sqrt{3x - 5} = \sqrt{3x + 8}$$

all three equations shown will be either negative roots which do not exist, or equations that cancel.

Practice:

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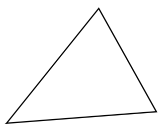
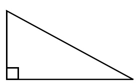
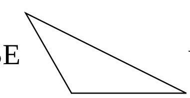
Section 8.4
The Pythagorean Theorem

Over four thousands years ago somewhere in China, or perhaps India, some say Egypt, a most remarkable tool for solving problems surfaced. Called *the pythagorean theorem*, after the 5th Century BCE Greek mathematician who proved it, it is a method of using a particular property of the right triangle to find the length of a missing side when the two other sides are known.

The angles of a triangle—a three-sided polygon—classify triangles as

ACUTE  when all angles are less than 90°

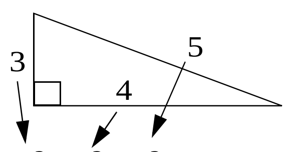
RIGHT  when one angle is 90°

OBTUSE  when one angle is greater than 90°

The Pythagorean Theorem concerns the right triangle (one angle 90°) only.

This theorem says that if we square all the sides of a triangle (this is important: ONLY when they are squared), the largest of the sides (the largest side is the one opposite the 90° angle) is equal to the sum of the other two sides.

Example:



$3^2 + 4^2 = 5^2$

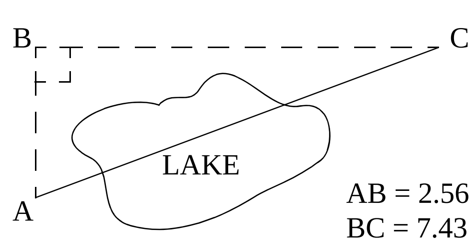
$9 + 16 = 25$

In this example, 3 has been squared, so has 4 and 5. If we add before squaring them, they don't "add up." Once squared, however, **3**, as **9**, and **4** as **16**, the sum is **25**, which is **5** squared.

The pythagorean theorem is very valuable, and it can be used, for example, to compute distances that can't be physically measured or for checking their accuracy.

Example: What is distance $\overline{AC}$ across the lake?

Measuring $\overline{AB}$ and $\overline{BC}$ was achieved over land; however, there is no need to measure $\overline{AC}$, for it can be computed using the pythagorean theorem.



AB = 2.56 miles
BC = 7.43 miles

Square $\overline{AB}$ $(2.56)^2 = 6.5536$ Square $\overline{BC}$ $(7.43)^2 = 55.2049$

Add $\overline{AB}$ and $\overline{BC}$ $6.5536 + 55.2049 = 61.7585$

Find the square root $\sqrt{61.7585} = 7.859$

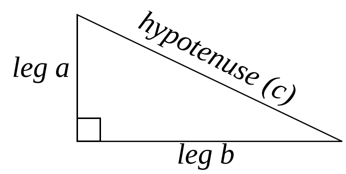
The distance across the lake is **7.859 miles** (rounded off)

THE PYTHAGOREAN FORMULA

The pythagorean theorem can be stated using the following formulas that compute each side:

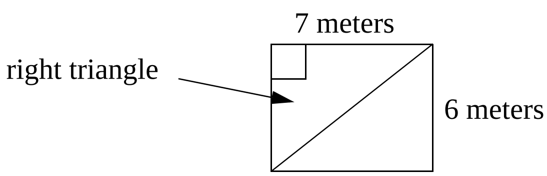
$$c = \sqrt{a^2 + b^2} \qquad a = \sqrt{c^2 - b^2} \qquad b = \sqrt{c^2 - a^2}$$

Called the *hypotenuse*, we'll make the largest side "***c***"
The other two sides we'll call leg "***a***" and leg "***b***."



Example: A carpenter is building the frame of a room. If the room is 6 meters wide and 7 meters long, how can she check if the room is accurately rectangular?

In a rectangle, the diagonals are equal. The best way for the carpenter to know if she has a rectangle with 90° angles, is to make the diagonals equal. What is the length of the diagonals if the sides are 6 and 7?



$$c = \sqrt{7^2 + 6^2} = \sqrt{49 + 36} = \sqrt{85} = 9.22$$

The diagonals must be **9.22 meters** for the room to be a rectangle.

Practice:

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Section 8.5
The Distance Formula

Using *x*-*y* graphical notation, the distance formula is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In a few words, the distance between two points is:

"The square root of the sum of the squares of the differences of the horizontal and vertical coordinates."

Example: What is the distance from point A(–34,–22) to point B(62,8)?

The formula is: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

horizontal distance — vertical distance

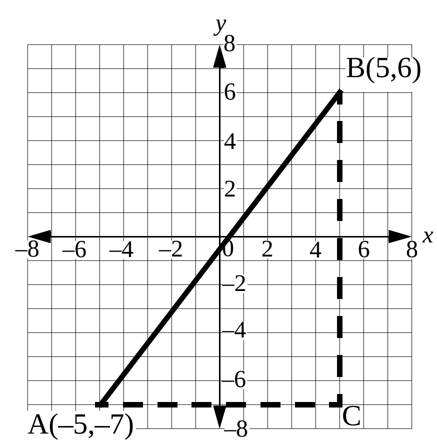
Where:

$x_1 = -34$
$x_2 = 62$
$y_1 = -22$
$y_2 = 8$

$$d = \sqrt{[62 - (-34)]^2 + [8 - (-22)]^2}$$
$$d = \sqrt{(62 + 34)^2 + (8 + 22)^2}$$
$$d = \sqrt{(96)^2 + (30)^2}$$
$$d = \sqrt{9216 + 900}$$
$$d = \sqrt{10116}$$
$$d = 100.6$$

The distance formula is an adaptation of the pythagorean theorem. In the graph below, $\overline{AB}$ is the hypotenuse of ΔABC, $\overline{BC}$ is the difference in *y* values, and $\overline{AC}$ is the difference of *x* values. In the graph you can count the distance from A to C (10), and from B to C (13). Using the pythagorean theorem:

$$AB = \sqrt{10^2 + 13^2} = \sqrt{100 + 169} = \sqrt{269} = 16.4$$



Using the distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$d = \sqrt{[5 - (-5)]^2 + [6 - (-7)]^2}$$
$$d = \sqrt{(5 + 5)^2 + (6 + 7)^2}$$
$$d = \sqrt{10^2 + 13^2}$$
$$d = \sqrt{100 + 169}$$
$$d = \sqrt{269}$$
$$d = 16.4 \quad \text{check!}$$

Practice: