Chapter 10

Rational Numbers

Section 10.1
Rational Expressions

Rational expressions are formed when we put polynomials in a fraction. In other words, rational expressions represent division.

Example: Simplify $\dfrac{x^2 + 3x - 10}{x^2 + x - 20}$

Because x^2, x, *10* and *20* are "tied up" in the trinomials through addition and subtraction, on first view it seems impossible that anything could be reduced, or simplified. However, the idea in rational expressions is to find ways to reduce the fraction by factoring. The above rational expression can be factored to:

$$\frac{(x+5)(x-2)}{(x-4)(x+5)}$$

The move is obvious. Now $\left(\dfrac{x+5}{x+5} = 1\right)$ can be cancelled and the rational expression becomes

$$\frac{x-2}{x-4}$$

Example: Simplify $\dfrac{7x + 14}{7x}$

$$\frac{7(x+2)}{7x}$$

$$\frac{x+2}{x}$$

Example: Simplify $\dfrac{y-5}{5-y}$

Because the expressions are not exactly the same, they cannot be cancelled. However, **5 – *y*** is the opposite of **y – 5** and it may be rewritten by factoring –1. Therefore,

$(5 - y)$ turns into $-(-5 + y)$ or $-(y - 5)$

And the new ratio is $\dfrac{y-5}{-(y-5)} = -1$ and can be reduced to –1.

Example: Simplify $\dfrac{x^2 - 9}{x^2 + 7x + 12}$

$$\frac{(x-3)(x+3)}{(x+4)(x+3)} = \frac{(x-3)}{(x+4)}$$

Example: Simplify $\dfrac{7 - 2c}{8(2c - 7)}$

Factor –1 from the numerator $\dfrac{-(-7 + 2c)}{8(2c - 7)} = \dfrac{-(2c - 7)}{8(2c - 7)} = -\dfrac{1}{8}$

Practice:

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Section 10.2
Multiplying Rational Expressions

Multiplication of rational expressions, like multiplication of fractions, is done by multiplying numerator with numerator and denominator with denominator, and reduction to lowest terms.

Example: Multiply $\frac{40}{x^4} \cdot \frac{5x^2}{8}$

combine using one fraction $\frac{40 \cdot 5x^2}{x^4 \cdot 8}$ and reduce $\frac{8 \cdot 5 \cdot 5 \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot 8} = \frac{25}{x^2}$

Example: Multiply $\frac{-3z}{9(z-1)} \cdot \frac{2z-2}{z^2}$

Factor 2 from $2(z-2)$ reduce 3, 9, z, and $(z-1)$ $\frac{(-3z)(2)(z-1)}{9(z-1)z^2} = \frac{-2}{3z}$

3 z

Example: Multiply $\frac{a^2-b^2}{2a+1} \cdot \frac{2a^2-5a-3}{a+b}$

Factor and reduce: Numerator left: Factor difference of two squares.
Numerator right: Factor trinomial (subtraction of two products).
Denominators cannot be factored.

reduce $\frac{(a+b)(a-b)(2a+1)(a-3)}{(2a+1)(a+b)}$ multiply $(a-b)(a-3) = a^2 - 3a - ab + 3b$

Practice:

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Section 10.3
Dividing Rational Expressions

Division of rational expressions, similar to the division of fractions, is done by multiplying the first fraction by the reciprocal of the second fraction. Once numerators and denominators are multiplied, reduce to lowest terms.

Example: Divide $\frac{-3a^6}{8} \div \frac{6a^2}{7}$ "Flip" (find reciprocal of) second fraction, reduce, and multiply what's left. $\frac{-3a^6 \cdot 7}{8 \cdot 6a^2}$

$$\frac{-3 \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot 7}{8 \cdot 2 \cdot 3 \cdot a \cdot a} = -\frac{7a^4}{16}$$

Example: Divide $\frac{-12 + 4z}{3} \div \frac{-6 + 2z}{6}$ Flip second fraction, factor, and reduce:

$$\frac{4(-3+z)}{3} \cdot \frac{6}{2(-3+z)} = \frac{4(-3+z) \cdot 6}{3 \cdot 2(-3+z)} = \frac{2 \cdot 2(-3+z) \cdot 2 \cdot 3}{3 \cdot 2 \cdot (-3+z)} = \frac{4}{1} = 4$$

Example: Divide $\frac{x^2 + 11x + 28}{x^2 + 5x - 14} \div \frac{-x^2 - 7x - 12}{x^2 + x - 6}$ Flip second fraction, factor and reduce:

$$\frac{(x+4)(x+7)}{(x+7)(x-2)} \cdot \frac{(x+3)(x-2)}{-(x+3)(x+4)} = -1$$

(–1 was factored out of the right denominator, making the trinomial positive, and the answer negative)

Practice:

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Section 10.4
Dividing Polynomials

DIVIDING BY A MONOMIAL

Example: Divide $\dfrac{8x^2 + 9x - 3}{4}$ **Each term** of the numerator is divided by 4

$$\frac{8x^2}{4} = 2x^2 \qquad \frac{9x}{4} = 2.25x \qquad -\frac{3}{4} = -0.75$$

Answer: $2x^2 + 2.25x - 0.75$

Example: Divide $\dfrac{9x^4 - 12x^3 - 14x}{3x}$

$$\frac{9x^4}{3x} = 3x^3 \qquad -\frac{12x^3}{3x} = -4x^2 \qquad \frac{-14x}{3x} = -4.\bar{6}$$

Answer: $3x^3 - 4x^2 - 4.\bar{6}$

DIVIDING BY A BINOMIAL

Division of a polynomial by a binomial is set up and computed similarly to regular division, except that here we use terms with base, coefficient, and exponent. In other words, each term will be divided, then multiplied and subtracted. Sometimes there will be a remainder, sometimes the last operation will leave none.

Example: Divide $(y^2 + 8y - 10) \div (y - 2)$

$$y - 2\overset{\displaystyle y}{\overline{\big)\, y^2 + 8y - 10}}$$

Divide the leading term of the trinomial by the leading term of the binomial: $\dfrac{y^2}{y}$ and place the result above the line like regular division.

$$\begin{array}{r} y \\ y - 2\overline{\big)\, y^2 + 8y - 10} \\ y^2 \qquad\qquad \end{array}$$

Like in regular division, multiply $y \cdot y$ and place it below the leading term of the trinomial.

$$\begin{array}{r} y \\ y - 2\overline{\big)\, y^2 + 8y - 10} \\ y^2 - 2y \qquad \end{array}$$

Do the same for the second term of the binomial, $y \cdot -2$

$$\begin{array}{r} y \\ y-2\overline{)\,y^2+8y-10} \\ -(y^2-2y) \\ \hline 10y \end{array}$$

Now subtract. The first term of the trinomial cancels and the second term becomes 10*y*. This is the end of the first cycle.

$$\begin{array}{r} y \\ y-2\overline{)\,y^2+8y-10} \\ -(y^2-2y)\downarrow \\ \hline 10y-10 \end{array}$$

To start the second cycle, like in regular division bring down –10 and place it next to 10*y*.

$$\begin{array}{r} y+10 \\ y-2\overline{)\,y^2+8y-10} \\ -(y^2-2y) \\ \hline 10y-10 \end{array}$$

Divide $\frac{10y}{y}$ and place answer above next to *y* and multiply ***y* – 2** by **10**.

$$\begin{array}{r} y+10 \\ y-2\overline{)\,y^2+8y-10} \\ -(y^2-2y) \\ \hline 10y-10 \\ -(10y-20) \\ \hline 10 \end{array}$$

Multiply the second term, 10, of the answer by both terms of the binomial (*y* – 2) and place under (10*y* – 10) and subtract.

remainder

Answer: $y + 10$ with $\frac{10}{y-2}$ remaining

Example: Divide $(x^3 + 1) \div (x + 1)$

$$x+1\overline{)\,x^3+1}$$

To properly divide an expression with missing terms like the binomial $x^3 + 1$, the middle terms x^2 and x must be inserted. This is done by leaving space for the missing terms before beginning.

$$x+1\overline{)\,x^3\qquad\qquad +1}$$

$$\begin{array}{r} x^2 \\ x+1\overline{)\,x^3\qquad\qquad +1} \\ -(x^3+x^2) \\ \hline -x^2 \end{array}$$

End of first cycle

$$\begin{array}{r} x^2-x \\ x+1\overline{)\,x^3\qquad\qquad +1} \\ -(x^3+x^2) \\ \hline -x^2 \\ -(-x^2-x) \\ \hline x \end{array}$$

End of second cycle

$$
\begin{array}{r l}
 & x^2 - x + 1 \quad \longleftarrow \text{Answer} \\
x + 1 \,\big) & x^3 \qquad\qquad + 1 \\
 & \underline{-(x^3 + x^2)} \\
 & \quad - x^2 \\
 & \quad \underline{-(-x^2 - x)} \\
 & \qquad\quad x + 1 \\
 & \qquad \underline{-(x + 1)} \quad \text{End of third cycle. No remainder}
\end{array}
$$

Practice:

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Section 10.5
Addition and Subtraction of Rational Expressions with Like Denominators

Adding and subtracting rational expressions is no different from regular addition or subtraction of fractions: There must be a common denominator.

ADDING AND SUBTRACTING WITH LIKE DENOMINATORS
When the denominators are the same, place all polynomials over the same denominator, then add or subtract while following the rules of addition and subtraction of polynomials.

Example: Add $\frac{8x-5}{3x}+\frac{7x-4}{3x}$

Because the denominators are identical, then

$$\frac{8x-5+7x-4}{3x}=\frac{15x-9}{3x}=\frac{3(5x-3)}{3x}=\frac{5x-3}{x}$$

Example: Subtract $\frac{2y+9}{y-3}-\frac{12y-5}{y-3}$

Because this is a subtraction, the ENTIRE second numerator must be written in parenthesis and subtracted:

$$\frac{2y+9-(12y-5)}{y-3}=\frac{2y+9-12y+5}{y-3}=\frac{-10y+14}{y-3}$$

Example: Add or subtract $\frac{x^2-x+12}{x^2-4}+\frac{x^2-8x-10}{x^2-4}-\frac{x^2-5x+14}{x^2-4}$

Write all trinomials over the same denominator. Because the third trinomial is subtracted, notice the sign changes.

$$\frac{x^2-x+12+x^2-8x-10-(x^2-5x+14)}{x^2-4}$$

$$\frac{x^2-x+12+x^2-8x-10-x^2+5x-14}{x^2-4}$$

Combine like terms and FACTOR to reduce.

$$\frac{x^2-4x-12}{x^2-4}=\frac{(x+2)(x-6)}{(x+2)(x-2)}=\frac{x-6}{x-2}$$

Example: Add and/or subtract $\frac{2x+5}{x^2-4x+4}+\frac{5x-2}{x^2-4x+4}-\frac{3x+7}{x^2-4x+4}$

Write all trinomials over the same denominator. Because the third trinomial is subtracted, notice the sign changes.

$$\frac{2x+5+5x-2-3x-7}{x^2-4x+4}$$

$$\frac{4x-4}{x^2-4x+4} \quad \text{factoring} \quad \longrightarrow \quad \frac{4(x-1)}{(x-2)(x-2)}$$

Factoring did not help to reduce. Answer remains $\frac{4x-4}{x^2-4x+4}$

Practice:

Section 10.6
Addition and Subtraction of Rational Expressions with Different Denominators

Fractions with different denominators cannot be added nor subtracted. To do so we must change them.

Example: Add $\frac{x+4}{3x} + \frac{x-4}{x}$

The denominators show x is common, but 3 is not. Correct by multiplying second fraction times $\frac{3}{3}$ (one).

$$\frac{x-4}{x} \times \frac{3}{3} = \frac{3x-12}{3x}$$

Complete addition using altered rational expression

$$\frac{x+4}{3x} + \frac{3x-12}{3x} = \frac{x+4+3x-12}{3x} = \frac{4x-8}{3x}$$

Example: Add $\frac{5}{y+2} + \frac{12}{y-2}$

The denominators are not the same, even if it is only a sign difference. The common denominator is $(y+2)(y-2)$.

Multiply both fractions by one using the common denominator $\frac{(y+2)(y-2)}{(y+2)(y-2)}$

$$\frac{5}{y+2} \times \frac{(y+2)(y-2)}{(y+2)(y-2)} = \frac{5(y-2)}{(y+2)(y-2)} \quad (1)$$

$$\frac{12}{y-2} \times \frac{(y+2)(y-2)}{(y+2)(y-2)} = \frac{12(y+2)}{(y+2)(y-2)} \quad (2)$$

Add the results (1) and (2)

$$\frac{5(y-2)+12(y+2)}{(y+2)(y-2)} = \frac{5y-10+12y+24}{(y+2)(y-2)}$$

$$= \frac{17y+14}{(y+2)(y-2)} = \frac{17y+14}{y^2-4}$$

Example: Subtract $\dfrac{x+5}{x^2+6x+9} - \dfrac{4}{x+3}$

Factoring the first denominator: $(x+3)(x+3)$

Because the second denominator is part of the second denominator, the first denominator is the *common denominator.*

Multiply the second fraction by **1** to get common denominator $(x+3)(x+3)$

$$\frac{4}{x+3} \times \frac{(x+3)}{(x+3)} = \frac{4x+12}{(x+3)(x+3)}$$

Subtract this result from the first fraction

write parenthesis for subtraction

$$\frac{x+5}{(x+3)(x+3)} - \frac{4x+12}{(x+3)(x+3)} = \frac{x+5-(4x+12)}{(x+3)(x+3)}$$

$$= \frac{x+5-4x-12}{(x+3)(x+3)}$$

$$= \frac{-3x-7}{(x+3)(x+3)} = \frac{-3x-7}{(x+3)^2}$$

Example: Subtract $\dfrac{-4}{x^2+6x+8} - \dfrac{-8}{x^2+8x+12}$

Factoring both denominators: $(x+2)(x+4)$ $(x+2)(x+6)$

Because $(x+2)$ is common to both, the common denominator is $(x+2)(x+4)(x+6)$

Multiply both fractions by **1** to get the common denominator

$$\frac{-4}{(x+2)(x+4)} \times \frac{(x+2)(x+4)(x+6)}{(x+2)(x+4)(x+6)} = \frac{-4(x+6)}{(x+2)(x+4)(x+6)}$$

$$\frac{-8}{(x+2)(x+6)} \times \frac{(x+2)(x+4)(x+6)}{(x+2)(x+4)(x+6)} = \frac{-8(x+4)}{(x+2)(x+4)(x+6)}$$

Subtract the results of the second fraction from the results of the first fraction:

$$\frac{-4(x+6)}{(x+2)(x+4)(x+6)} - \frac{-8(x+4)}{(x+2)(x+4)(x+6)} = \frac{-4(x+6)-[(-8)(x+4)]}{(x+2)(x+4)(x+6)}$$

(*Continues in next page*)

$$= \frac{-4x - 24 + 8x + 32}{(x+2)(x+4)(x+6)}$$

$$= \frac{4x + 8}{(x+2)(x+4)(x+6)}$$

$$= \frac{4(x+2)}{(x+2)(x+4)(x+6)}$$

$$= \frac{4}{(x+4)(x+6)} = \frac{4}{x^2 + 10x + 24}$$

Practice:

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Section 10.7
Rational Equations

A rational equation is one that contains fractions. Because fractions are divisions, one simple way to start the solution of rational equations is by the use of multiplication. This will make for a less complicated solution as the fractions are eliminated.

Example: Solve $\frac{2}{5}+\frac{7}{2} = \frac{y}{10}$

The denominators show 10 is the common denominator for the equation; therefore, **multiplying each term** of the equation by 10 will eliminate the rational numbers (fractions).

$\frac{(10)2}{5}+\frac{(10)7}{2} = \frac{(10)y}{10}$ reducing each term: $4+35 = y$ ➡ $y = 39$

Example: Solve $y-\frac{5}{y} = -4$

This is a quadratic equation: Write equal to zero, factor, and solve.

Multiply every term by the common denominator y: $(y)y-\frac{(y)5}{y} = (y){-4}$ ➡ $y^2-5 = -4y$

solve for y: $y^2-5 = -4y$ ➡ $y^2+4y-5 = 0$ ➡ $(y+5)(y-1) = 0$

Answer: $y = -5, 1$

Example: Solve $\frac{5}{y+2} = \frac{12}{y-2}$

The denominators are not the same, even if it is only a sign difference. The common denominator is $(y+2)(y-2)$. Multiply both sides by $(y+2)(y-2)$.

$$\frac{(y+2)(y-2)5}{y+2} = \frac{(y+2)(y-2)12}{y-2}$$

reduce to: $(y-2)5 = (y+2)12$

distribute: $5y-10 = 12y+24$

solve for y: $-10-24 = 12y-5y$

$-34 = 7y$

$y = -\frac{34}{7}$

Example: Solve $\dfrac{1}{(x+5)} + \dfrac{3}{(x-5)} = \dfrac{x^2-9}{x^2-25}$

The common denominator is $(x + 5)(x - 5)$, which happens to be also the difference of squares $x^2 - 25$. Multiply all three terms by $(x + 5)(x - 5)$ and reduce:

$$\frac{(x+5)(x-5)1}{(x+5)} + \frac{(x+5)(x-5)3}{(x-5)} = \frac{(x+5)(x-5)x^2-9}{x^2-25} \qquad (x-5)1 + (x+5)3 = x^2-9$$

distribute: $x - 5 + 3x + 15 = x^2 - 9$ combine: $0 = x^2 - 4x - 19$

Solve for x by completing the square: $x^2 - 4x + 4 = 19 + 4$ ➡ $\sqrt{(x-2)^2} = \sqrt{23}$

$x - 2 = \sqrt{23}$ ➡ $x = 2 \pm \sqrt{23}$

Practice:
Solve.

1. $\dfrac{3}{4} + \dfrac{1}{3} = \dfrac{x}{12}$
2. $\dfrac{4}{9} + \dfrac{2x}{6} = \dfrac{x}{3}$
3. $\dfrac{2}{5x} + \dfrac{7}{x} = 1$
4. $\dfrac{8a}{5} + \dfrac{12a}{20} = \dfrac{a}{4}$
5. $\dfrac{5}{4a} + \dfrac{1}{3} = \dfrac{4}{9}$
6. $\dfrac{7}{6x} + \dfrac{1}{5x} = \dfrac{1}{15}$
7. $\dfrac{2}{x-1} + \dfrac{7}{x+1} = 1$
8. $\dfrac{x+1}{4} + \dfrac{x+2}{5} = 1$
9. $\dfrac{y-1}{3} + \dfrac{y-2}{2} = \dfrac{3}{4}$
10. $\dfrac{y+2}{7} - \dfrac{y-5}{14} = \dfrac{1}{2}$
11. $\dfrac{1}{y^2-9} - \dfrac{5}{(y-3)} = 1$
12. $z - \dfrac{12}{z} = \dfrac{1}{z}$
13. $\dfrac{y-3}{y+4} = \dfrac{y-5}{y+1}$
14. $\dfrac{2y-5}{y+3} = \dfrac{12}{y-8}$
15. $\dfrac{7a-1}{3a+5} = \dfrac{4a+3}{3a-5}$
16. $\dfrac{7}{x-3} - \dfrac{8}{x+3} = \dfrac{-4}{5}$
17. $\dfrac{y}{8} - \dfrac{5y}{24} = \dfrac{1}{6}$
18. $\dfrac{14}{x+8} - \dfrac{3}{x-4} = 1$
19. $\dfrac{15}{y+1} = \dfrac{14}{y-7}$
20. $\dfrac{5}{4(y+1)} = \dfrac{12}{2(y+1)}$
21. $\dfrac{24}{9(c-2)} = \dfrac{2c}{18}$
22. $\dfrac{5(y+3)}{3y+1} = \dfrac{2(y-5)}{3y-1}$
23. $\dfrac{p-3}{2p-1} = \dfrac{p+10}{2p+1}$
24. $\dfrac{-4}{z-1} + \dfrac{-z-5}{z+1} = 1$
25. $\dfrac{8}{x-4} + \dfrac{12}{x+4} = 5$
26. $\dfrac{1}{x-8} - \dfrac{1}{x+8} = \dfrac{-1}{2}$
27. $\dfrac{(y+5)}{4y+3} = \dfrac{(y-8)}{4y-3}$
28. $\dfrac{7}{x+1} - \dfrac{14}{x-1} = \dfrac{x^2-4}{x^2-1}$
29. $\dfrac{12}{2x-7} + \dfrac{6}{2x+7} = \dfrac{4x-9}{4x^2-49}$
30. $\dfrac{16}{x+8} - \dfrac{11}{x-8} = \dfrac{x^2+25}{x^2-64}$
31. $\dfrac{15}{x+4} + \dfrac{60}{x^2-16} = \dfrac{x+2}{x-4}$
32. $\dfrac{10}{2x+3} + \dfrac{9}{2x-3} = 1$
33. $\dfrac{18}{x^2-49} - \dfrac{20}{x-7} = \dfrac{1}{x+7}$
34. $\dfrac{7}{x^2-x-2} = \dfrac{8}{x^2-2x-3}$
35. $\dfrac{x-2}{x^2+5x+4} = \dfrac{x+3}{x^2+6x+8}$

Section 10.8
Complex Fractions

This a complex fraction: $$\frac{8+\frac{1}{x-1}}{\frac{5}{x+1}-2}$$

Basically, these are large fractions made from fractions. To solve them, approach one section at a time.

THE NUMERATOR

The numerator of the "mother" fraction is an addition of fractions that reads: $\frac{8}{1}+\frac{1}{x-1}$

where the common denominator is $(x-1)$

Multiply left-hand fraction to adjust for the common denominator $\frac{8}{1}\times\frac{(x-1)}{(x-1)} = \frac{8x-8}{x-1}$

Add numerator fractions: $\frac{8x-8}{x-1}+\frac{1}{x-1} = \frac{8x-8+1}{x-1} = \frac{8x-7}{x-1}$ (**1**)

THE DENOMINATOR

These denominator fractions are performing subtraction $\frac{5}{x+1}-\frac{2}{1}$

Multiply right-hand fraction to adjust for the common denominator: $\frac{2}{1}\times\frac{(x+1)}{(x+1)} = \frac{2x+2}{x+1}$

Subtract denominator fractions: $\frac{5}{x+1}-\frac{2x+2}{x+1} = \frac{5-2x-2}{x+1} = \frac{-2x+3}{x+1}$ (**2**)

Dividing numerator and denominator

Divide "mother" fraction (results from numerator (**1**) over results of denominator (**2**): $\frac{8x-7}{x-1}\div\frac{-2x+3}{x+1}$

multiply by the reciprocal $\frac{8x-7}{x-1}\times\frac{x+1}{-2x+3} = \frac{(8x-7)(x+1)}{(x-1)(-2x+3)} = \frac{8x^2+x-7}{-2x^2+5x-3}$

Example: simplify $$\frac{\frac{2a}{5}+\frac{4a}{5}}{\frac{3a}{8}}$$

Numerator first: $\frac{2a}{5} + \frac{4a}{5} = \frac{2a + 4a}{5} = \frac{6a}{5}$

Divide results of numerator by denominator: $\frac{6a}{5} \div \frac{3a}{8} = \frac{6a}{5} \times \frac{8}{3a} = \frac{48a}{15a} = \frac{16}{5}$

Example: Simplify $\dfrac{3 - \frac{1}{y}}{9 - \frac{1}{y^2}}$

Numerator first: $\frac{3}{1} - \frac{1}{y} = \frac{3y - 1}{y}$ Then denominator: $\frac{9}{1} - \frac{1}{y^2} = \frac{9y^2 - 1}{y^2}$

Divide numerator by denominator. (Notice factoring of difference of squares).

$$\frac{3y-1}{y} \div \frac{9y^2-1}{y^2} = \frac{3y-1}{y} \times \frac{y^2}{9y^2-1} = \frac{y^2(3y-1)}{y(3y+1)(3y-1)} = \frac{y}{(3y+1)}$$

Practice:

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