Chapter 1

Working with Numbers and Variables

Section 1.1
Turning Plain English into Algebra

In algebra, simple words can be turned into algebraic expressions. Expressions that define mathematical relationships between variables.

Examples:

PLAIN ENGLISH	ALGEBRA
1. The **product** of five and six is thirty.	$5 \times 6 = 30$ (product is the result of multiplication)
2. The **quotient** of thirty and five is six.	$\frac{30}{5} = 6$ (quotient is the result of division)
3. The **sum** of three and four is seven.	$3 + 4 = 7$ (sum is the result of addition)
4. The **difference** of twelve and four is eight.	$12 - 4 = 8$ (difference is the result of subtraction)
5. A number	any letter will do (*a, b, c, d,.....w, x, y, z*)
6. Twice a number	$2x$
7. Four times a number	$4a$
8. Three consecutive numbers	$x, (x+1), (x+2)$
9. If seven is added to a number...	$x + 7$ *or* $7 + x$
10. If ten is subtracted from a number...	$n - 10$ (backwards has a different meaning)
11. Eight less than twice a number	$2a - 8$
12. Five more than three times a number	$3y + 5$
13. A number to the fifth power plus three	$x^5 + 3$
14. Ten increased by half a number.	$10 + \frac{x}{2}$
15. The product of two numbers	xy
16. The square root of a number decreased by four.	$\sqrt{n} - 4$
17. Pete **is** three years older than twice the age of Nick.	$P = 2N + 3$ ("is" stands for equal)
18. Twice the sum of two numbers	$2(a + b)$
19. Seven times the difference of two numbers	$7(c - d)$
20. Three-fourth a number reduced by six.	$\frac{3x}{4} - 6$

21. Five is to six as ten is to twelve (*proportion*).	$\frac{5}{6} = \frac{10}{12}$
22. The sum of two numbers, squared.	$(a + b)^2$
23. The sum of two numbers squared.	$a^2 + b^2$
24. Mount Everest is higher than Pico Aconcagua.	$E > A$
25. David's score is at least as high as Pedro's.	$David \geq Pedro$
26. Sandra earns less than Francis.	$Sandra < Francis$

Practice:
Write the equivalent algebraic expression.

1. Lesly is fourteen inches taller than Jim.
2. The product of a number and the same number.
3. Half a number, increased by twelve.
4. The difference between two numbers.
5. The difference of two squares.
6. Three times a number, reduced by six.
7. Jerry is three times Ralph's age.
8. The sum of two consecutive numbers.
9. Half a number, increased by seven.
10. The addition of two squared numbers.
11. Felix is five years younger than Mimi.
12. In three years, Juan will be three times as old as Elaine.
13. The boy has quarters and dimes.
14. The difference between A and B is 500.
15. Twice a number, plus three times another number.
16. The girl is half her mother's age.
17. Petra is 30 yards ahead of Jim.
18. The difference between C and D is fifteen.
19. The combined ages of Ed and Al is fifty six.
20. Four times a number reduced by eight.
21. Twice the sum of two numbers.
22. Half the area of a circle.
23. Three times the difference of two numbers.
24. Twice a number, less the number increased by two.
25. Three times a number increased by one-third the number.
26. One third a number, increased by nine.
27. His grandmother is four times his age.
28. A centimeter is larger than a millimeter.
29. Half a number is larger than one-third a number.
30. Jack is at least as tall as Holly.
31. Fred earns twice as much as Jordi.
32. The quotient of two numbers is five.
33. Twice a number, increased by seven, is five.
34. The product of a number and its square root is nine.
35. The ratio of two numbers is ten.
36. The square of a number increased by the number.
37. The cube of a number reduced by the square of the number.
38. Laura is shorter than David and David is shorter than Mack.
39. The baby is at least twenty-one inches long.
40. Three times the sum of three numbers.
41. Four times a number plus the product of the number and five.
42. The difference of two numbers, squared.
43. The square root of the sum of two numbers squared.
44. The product of two consecutive numbers.
45. Twice a number, increased by six.
46. The square root of a number, increased by the cube of the number.
47. The square root of twice the number.

Write the equivalent English sentence.

1. $x + y + z$
2. $a - b$
3. $2x - 7$
4. $\frac{c}{2} - 10$
5. $t > s$
6. $Mary + 3 = Susan$
7. $a - b = 8$
8. $2(x + y)$
9. $x + (x + 2) + (x + 4)$
10. $3x + 2x$
11. $4(a - b)$
12. $n + 8$
13. abc
14. $6x + 5$
15. $33 + 34 + 35 = 102$
16. $\sqrt{x} + 5$
17. $x^2 + y^2$
18. $(a^2 - b^2)$
19. $4g$
20. $\frac{x}{3} + 6$
21. $5 + d$
22. $a \geq b$
23. $c \leq d$
24. $3x + 4y - 7z$
25. $x(x - 4)$
26. $5N + 13D + 10Q = \$4.05$
27. $\angle A = \angle B$
28. $\frac{x + y}{2}$
29. $David > John > Carlos$
30. $x^3 - 17$
31. $8 - y$
32. $a + b - c$
33. $x - 5$
34. $4y + 4$
35. $\frac{b}{3} + 12$
36. $m < n$
37. $Pete = Joan + 10$
38. $a + b + c$
39. $x - y$
40. $4x - 8$
41. $\frac{x}{6} - 5$
42. $c > b$
43. $Carlos + 14 = Mark$
44. $m + n = 6$
45. $5(x - y)$
46. $x + (x + 1) + (x + 2)$
47. $4x - 6x$
48. $5(x - y)$
49. $n - 12$
50. xyz
51. $2x + 15$
52. $21 + 22 + 23$
53. $\sqrt{b} + 7$
54. $x^3 + y^3$
55. $(a^3 - b^3)$
56. $7f$
57. $\frac{x}{4} + 8$
58. $15 - p$
59. $a \leq -9$
60. $12 \geq j$
61. $5x + 3y + 8z$
62. $(y - 12)y$
63. $15N + D + 20Q = \$5.85$
64. $\angle 2F = \angle T$
65. $\frac{a - b}{4}$
66. $David < John < Carlos$
67. $x^5 + 34$
68. $\frac{c}{4} + 8$
69. $c > d$
70. $Jerry + Paco = 40$

Section 1.2
Order of Operations

Order of Operations is the sequence of steps that must be taken to reduce a math expression to its simplest form, or scale down an equation to find an answer. Not following the proper order will result in the wrong answer. Understanding these rules is the key to success in algebra and beyond.

STEPS:
1. First, simplify all expressions **inside** all parentheses. If there are parentheses inside the parentheses [or brackets], work from the inside parentheses first, eliminating parentheses as you go along.
2. Next, do exponents and radicals (roots).
3. Then, do multiplication and division from left to right as they occur. Multiplication is noted with an ×, dot $(2 \cdot 3)$, or parentheses, like in 3(4) = 12. Division is noted with a slash (/), fraction line (–), or ÷.
4. Lastly, do addition and subtraction.

Example: $3 + 4 \times 8 = 35$ FIRST multiply 4 × 8, then add 3

If you want to add first, place the addition in parenthesis: $(3 + 4) \times 8 = 56$

Add 3 and 4, then multiply by 8. Notice the different answers.

Example:

$7 + 2 \times 5\,(5 + 6^2) - 3 \times 4$	(exponent)
$7 + 2 \times 5\,(5 + 36) - 3 \times 4$	(parenthesis)
$7 + 2 \times 5\,(41) - 3 \times 4$	(multiplication)
$7 + 2 \times 205 - 12$	(multiplication)
$7 + 410 - 12 = 405$	(add and subtract)

Example:

$9 + 48 \div [2\,(4 + 2^3)] - 15 \div 3 \times 2$	(exponent)
$9 + 48 \div [2\,(4 + 8)] - 15 \div 3 \times 2$	(parenthesis)
$9 + 48 \div [2\,(12)] - 15 \div 3 \times 2$	(parenthesis and bracket)
$9 + 48 \div 24 - 15 \div 3 \times 2$	(divide and multiply)
$9 + 2 - 10 = 1$	(add and subtract)

Example:

$3[22 + 36 \div 2\,(10 - \sqrt{16}\,) + (7 - 2^2)]$	(root and exponent)
$3[22 + 36 \div 2\,(10 - 4) + (7 - 4)]$	(parentheses)
$3[22 + 36 \div 2\,(6) + 3]$	(division)
$3[22 + 18(6) + 3]$	(multiplication)
$3[22 + 108 + 3]$	(addition)
$3[133] = 399$	(multiplication)

Practice:
Evaluate.

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Section 1.3
Arithmetic and Geometric Sequences

A sequence is a series of numbers that follow a fixed pattern. A sequence can be arithmetic or geometric. Arithmetic sequences use the ***addition*** of a particular number; geometric sequences use the ***multiplication*** of a particular number. For example, the series 3, 6, 9, 12... is arithmetic and the series 3, 9, 27, 81 is geometric.

Finding a Missing Number in an Arithmetic Sequence

— First, determine if the sequence is increasing or decreasing.

— Secondly, find the difference between the first two numbers, the second and third number, the third and fourth number, and so on. If the difference is always the same (fixed), then there is a sequence. Continuing the sequence requires that the difference found is added (or subtracted) continuously.

Finding a Missing Number in a Geometric Sequence

— The multiplier by which the sequence is increasing (or decreasing) must be identified first. If the sequence increases (or decreases) by the same **factor**, then it is a geometric sequence. Continuing the sequence requires that each number be the product of the LAST number TIMES the factor by which the sequence is changing.

Example:

Find the missing number in the sequence. Determine if the sequence is arithmetic, geometric, or neither of the two.

4, 9, 14, ..., 24, 29...

The difference between the first and second number is 5 (9 – 4 = 5), between second and third is also 5 (14 – 9 = 5); therefore, the sequence is increasing by the same number, 5, and the sequence is arithmetic. The missing number is 19 (14 + 5 = 19). After 29 the next number is 34 (29 + 5 = 34), then 39, and so on.

Example:

Find the missing number in the sequence. Determine if the sequence is arithmetic, geometric or neither of the two.

...4, 7, 11, 16, ..., 29, 37...

The differences begin with three (7 – 3 = 4), then four (11 – 4 = 7), five (16 – 5 = 11)... therefore, the missing number is found by adding six to 16 to get 22 and seven to 22 to get 29, and so on. Because the next number is NOT found by adding the same FIXED amount, then the sequence is neither arithmetic nor geometric.

Example:

Find the missing number in the sequence. Determine if the sequence is arithmetic, geometric or neither of the two.

1, 3, 9, 27, ..., 243...

Because the differences do not fit an addition pattern (the differences are 2, 6, 18), the sequence is not arithmetic; however, the numbers do increase by a factor of 3 (1 × 3 = 3, 3 × 3 = 9, 9 × 3 = 27); therefore, the missing number is 81 (27 × 3 = 81) and the sequence is geometric.

Practice:

In the following exercises, determine if the sequence is arithmetic, geometric, or neither of the two. Then select the next two numbers that represent the sequence, if a sequence is found.

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Section 1.4
The Properties of Algebra

COMMUTATIVE PROPERTY

Have you ever noticed that we CAN add any number in any order we want, but we CAN'T do the same with subtraction? Also, we can multiply in any order we want, but not in division.

Example:

We can add $3 + 5 + 2 = 10$
and switch the order to $5 + 3 + 2$
and still get the same answer.

Or multiply $8 \times 6 \times 4 = 192$
and re-arrange to $6 \times 4 \times 8$
and still get the same answer.

If you try the same with subtraction and division, it will not work. This is what is called the **commutative** property of addition and multiplication: Changing the order of the numbers will not affect the answer. The word *commutative* means exchange.

ASSOCIATIVE PROPERTY

Another neat truth about addition and multiplication is that you can "**associate**" as many numbers as you want and still get the same answer. This association is done by the use of parentheses.

Example:

$(12 + 8) + 14 = 12 + (8 + 14)$
Whether you add 12 and 8 first, or 8 and 14 first, the answer will not change.

$7(18 \times 2) = (7 \times 18)\,2$
Whether you multiply 18 and 2 first, or 7 and 18 first, the answer will not change.

DISTRIBUTIVE PROPERTY

The example that follows tells you that you can add first and then multiply, or you can multiply first and then add. It doesn't matter; the answer will always be the same. Because the number outside the parenthesis is distributed over all the numbers inside the parenthesis, we call this a "**distributive**" property.

Example:

add first, then multiply
$3(9 + 4 + 6) = 3\,(19) = 57$

multiply first, then add
$3(9 + 4 + 6) = 3(9) + 3(4) + 3(6) = 27 + 12 + 18 = 57$

Although according to "order of operations" rules you are supposed to do what is inside the parenthesis first, sometimes this is not possible. For example in

$$5(x + y + z) = 5x + 5y + 5z$$

x, y, and z cannot be added without knowing what x, y, and z represent. Therefore, "distribute" 5 over the THREE terms of the trinomial $(x + y + z)$

INVERSE PROPERTY

Inverse properties pair the arithmetic operations we use in mathematics. "Inversion" means that pairs of numbers do the opposite—they "undo" each other.

For example, when subtraction "undoes" addition, we call it the **additive inverse** (negative) property. In this case addition (+) and subtraction (–) are being paired.

When division "undoes" multiplication, it is called the **multiplicative inverse** (reciprocal) property. In this case multiplication (×) and division (÷) are being paired.

In algebra, we also pair the "square" (raising to the **second** power: $x \cdot x = x^2$, $3 \cdot 3 = 3^2 = 9$) with the "square root" ($\sqrt{9} = 3$). In other words, the *square* and the *square root* are opposite of each other.

Examples:

Addition—Subtraction $8 - 8 = 0$ $x - x = 0$

Multiplication—Division $5 \cdot \frac{1}{5} = \frac{1}{5} \cdot 5 = 1$ $x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$

Square—Square Root $4^2 = 16$ $\sqrt{16} = 4$ $x^2 = (x)(x)$ $\sqrt{x^2} = \sqrt{(x)(x)} = x$

IDENTITY PROPERTY

The identity property also has to do with addition and multiplication. If we add **zero** to any number, the answer is the number; if we multiply **one** by any number, the answer is the number.

The "**additive identity**" is: $3 + 0 = 3$ or $a + 0 = a$

And the "**multiplicative identity**" is: $7 \times 1 = 7$ or $a\,(1) = a$

Practice:

Find the algebraic property being represented.

1. $5\,(3 + 4 + 7) = 5(3) + 5(4) + 5(7)$
2. $19 \times 1 = 19$
3. $\sqrt{36} = 6$
4. $12\,(5) + 12\,(8) = 12\,(5 + 8)$
5. $1 \times 12 = 12$
6. $(5 + 83) + 4 = 5 + (83 + 4)$
7. $18 + 3 + 12 + 7 = 12 + 3 + 7 + 18$
8. $-7 + 7 = 0$
9. $13 \times 16 \times 41 \times 8 = 41 \times 16 \times 8 \times 13$
10. $43 + 0 = 43$
11. $(17 \times 18) \times 9 = 17 \times (18 \times 9)$
12. $12(2) + 12(9) + 12(10) = 12\,(2 + 9 + 10)$
13. $44 + 66 + 77 = 66 + 77 + 44$
14. $0 = 15 - 15$
15. $10 - 10 = 0$
16. $\frac{1}{12} \cdot 12 = 1$
17. $34 \times 1 = 34$
18. $9\,(13 + 14 + 17) = 9(13) + 9(14) + 9(17)$
19. $10 + 0 + 54 + 13 = 13 + 10 + 54 + 0$
20. $(22 + 13) + 21 = 22 + (13 + 21)$
21. $16 \times 43 \times 22 = 22 \times 16 \times 43$
22. $0 + 17 = 17$
23. $(15 \times 23) \times 41 = 15 \times (23 \times 41)$
24. $6(16) + 6(9) + 6(11) = 6\,(16 + 9 + 11)$
25. $23(2 + 5 + 88) = 23(2) + 23(5) + 23(88)$
26. $\sqrt{81} = 9$
27. $12 + 0 = 12$
28. $3(15) + 3(18) + 3\,(-33) = 3\,(15 + 18 - 33)$
29. $1 \times 32 = 32$

Section 1.5
Ratios, Proportion, and Percent

RATIOS AND PROPORTION

A **ratio**, also called a fraction, is a comparison between two numbers using division.

Example: A car is travelling at 55 miles PER hour.

This is a ratio expressed in $\frac{miles}{hour}$ (the word PER takes the place of the fraction line)

Example: The price of rice is $47 for 25 pounds. $\frac{DOLLARS}{POUND} = \frac{47}{25}$

Example: The pitcher is allowing 2.75 runs per 9 innings. $\frac{RUNS}{9\ \ INNINGS} = \frac{2.75}{9}$

A **proportion** is an arithmetic sentence stating that *two ratios are equal*. $\frac{a}{b} = \frac{c}{d}$

Because proportions contain four numbers, proportions are used to solve ratio problems where one of the four numbers is missing. Any of these four combinations are solved using cross multiplication and division, in that order. So that:

$\frac{a}{b} = \frac{c}{d}$ is $\boldsymbol{ad} = \boldsymbol{bc}$ and $a = \frac{bc}{d}$ $b = \frac{ad}{c}$ $c = \frac{ad}{b}$ $d = \frac{bc}{a}$

Simply stated: To find any **unknown** in a proportion, multiply across and divide by the third number. In the case of (***a***) above, multiply (***b***)(***c***) and divide the result by (***d***); for (***b***), multiply (***a***)(***d***) and divide by (***c***); for (***c***), multiply (***a***)(***d***) and divide by (***b***); for (***d***), multiply (***b***)(***c***) and divide by (***a***).

Example: A person in an automobile travels 260 miles in 4.5 hours. At this rate, how far can she travel in a period of 9 hours?

She travels in miles per hour $\frac{miles}{hour}$ or $\frac{260}{4.5}$ or how many miles in nine hours $\frac{miles}{9}$

$$\frac{260}{4.5} = \frac{m}{9}$$

Multiply across (260)(9) = 2340. Divide the results by 4.5. $\frac{2340}{4.5} = 520$ miles

To check, include the answer in the proportion $\frac{260}{4.5} = \frac{520}{9}$ and cross multiply 2340 = 2340

Example: A soldier brings home a recipe that requires 60 pounds of butter to make 1,100 cookies. How many pounds of butter will he need at home if he only wants to make 40 cookies?

The units involved are pounds of butter and number of cookies, or how much **butter** per **cookie**: $\frac{B}{C}$

$$\frac{B}{C} = \frac{60}{1100} = \frac{B}{40}$$

Multiply and divide: $B = \frac{60 \times 40}{1100} = \frac{2400}{1100} = 2.18$ pounds of butter.

PERCENT

Percent problems are also proportional problems, where a percent ratio $\left(\frac{x}{100}\right)$ is always equal to a known ratio.

Example: A shirt advertised for $40 is reduced $8 and you buy it for $32. What percent of the original price did you SAVE?

$$\frac{SAVE}{PRICE} = \frac{8}{40} = \frac{S}{100}$$

Multiply and divide: $\frac{8 \times 100}{40} = \frac{800}{40} = 20\,\%$ Answer: 20% savings

Example: If the sales tax rate in a certain state is 7%, how much sales tax must you pay for an item that costs $45?

$$\frac{TAX}{COST} = \frac{T}{45} = \frac{7}{100}$$

Multiply and divide: $\frac{45 \times 7}{100} = \frac{315}{100} = 3.15$ Answer: $3.15

Example: A certain bank requires a 15% down-payment for a house. If you have $63,750 for a down-payment, what is the highest price you could pay for a house?

$$\frac{DOWN\ \ PAYMENT}{PRICE} = \frac{63750}{P} = \frac{15}{100}$$

$\frac{63750 \times 100}{15} = \frac{6375000}{15} = 425,000$ The price cannot exceed $425,000.

Practice:

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Section 1.6
The Number Line

UNDERSTANDING NUMBERS

All the numbers that you have used so far in mathematics to compute operations and solve problems are called *real numbers*. A real number could be any number, fraction or decimal.

Examples: The following are all real numbers:

$0.\overline{1}$ $\qquad \frac{1}{2}$ $\qquad \sqrt{5}$ $\qquad \pi$ $\qquad 6.524$ $\qquad 40$ $\qquad 10{,}000$

Real numbers may be broken down into *Irrational Numbers* and *Rational Numbers*.

Real Numbers → Irrational Numbers (non-repeating decimals like $\sqrt{5}$, π)

Real Numbers → Rational → Fractions and Repeating Decimals ($\frac{1}{2}$, $0.\overline{1}$, $4.\overline{325}$)

Real Numbers → Rational → Integers (...–1, 0, 1...) → Whole (0, 1, 2...) → Natural (1, 2, 3...)

Irrational Numbers

Irrational numbers are those numbers that cannot be written as a *ratio* of two integers.

Examples: The following are irrational numbers: $\sqrt{2}$ $\qquad \pi$ $\qquad \sqrt{10}$ $\qquad \sqrt{500}$

Rational Numbers

A number is **ratio**nal if it can be written as a *ratio* of two integers, a fraction.

Examples: The following are rational numbers:

$0.\overline{3}$ $\qquad \frac{3}{4}$ $\qquad \frac{2}{1}$ $\qquad \frac{17}{4}$ $\qquad 8.\overline{4532}$ $\qquad 10.3$ $\qquad \sqrt{169}$

Repeating decimals and square roots of *perfect squares* are also rational numbers.

UNDERSTANDING INTEGERS

Rational numbers not written as a ratio are called *Integers*. Integers can be positive or negative and they include *zero*. Integers can be best understood by the use of the *Number Line*.

– +

–9 –8 –7 –6 –5 –4 –3 –2 –1 0 1 2 3 4 5 6 7 8 9

Number lines have a *zero* in the middle, *negative* numbers to the left, and *positive* numbers to the right. To use the number line to find answers, start at zero and then move left and right according to whether the expression represents an addition or subtraction. Where the last operation lands, it gives the answer.

Example: Evaluate $-3 + 5 + 7 - 9 + 4 - 6 + 2 + 8 - 15$

Starting at zero move 3 to the left (because it is negative 3), 5 to the right, 7 right, 9 left, 4 right, 6 left, 2 right, 8 right, 15 left = –7

Adding Integers

Using the number line and adding positive integers gives a positive answer: $45 + 15 = 60$

Using the number line and adding negative integers gives a negative answer. $-20 + (-13) = -33$

Using the number line and adding a negative and a positive number gives an answer that could be either positive or negative, depending on whether the larger number is positive or negative:

$$+14 + (-8) = 6 \qquad +12 + (-19) = -7$$
$$14 - 8 = 6 \qquad 12 - 19 = -7$$

Subtracting Integers

Subtracting integers is also called “algebraic subtraction”, which involves “taking away.”

Taking away a positive number: $15 - (8) = 7$ $\quad -15 - (8) = -23$
$15 - (+8) = 7$ $\quad -15 - (+8) = -23$

Examples:

Jerry has \$15 in his pockets and “takes away” \$8 to buy lunch: 15 – (8) = \$7 left.

Jerry uses his credit card (he owes) to spend \$15 for groceries and \$8 for lunch:

$-15 - (8) = -23$ (owes \$23)

Taking away a negative number: $25 - (-10) = 35$ $\quad -25 - (-10) = -15$
$25 + 10 = 35$ $\quad -25 + 10 = -15$

Examples:

Jerry has \$25 left after paying \$10. If the payment is returned: 25 – (–10) = \$35

On credit, Jerry buy \$25 in groceries. If \$10 are returned: –25 – (–10) = –15

Because there is a benefit from taking away a negative, the product of negative × negative is positive.

Practice:

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